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A COMPUTATIONALLY ATTRACTIVE BEAM THEORY ACCOUNTING FOR TRANSVERSE SHEAR AND NORMAL DEFORMATIONS

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MECHANICS AND STRUCTURES BRANCH

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ABSTRACT

A variational higher-order theory has been developed for representing the bending and stretching of linearly elastic orthotropic beams which include the deformations due to transverse shearing and stretching of the transverse normal. The theory assumes a linear distribution for the longitudinal displacement and a parabolic variation of the transverse displacement across the thickness. Independent expansions are also introduced in order to represent the through-thickness displacement gradients by requiring least-square compatibility for the transverse strains and the exact stress boundary conditions at the top/bottom beam surfaces. The theory is shown to be well suited for finite element development by requiring simple C^0- and C^1-continuous displacement interpolation fields. Computational utility of the theory is demonstrated by formulating a simple two-node stretching-bending finite element. Both analytic and finite element procedures are applied to a simple bending problem and compared to an exact elasticity solution. It is shown that the inclusion of the transverse normal deformation in the present theory provides an improved displacement, strain and stress prediction capability, particularly for the analysis of thick-section beams.

NOMENCLATURE

A	cross-sectional area of beam
A_{ij}	inplane rigidities
b	width of beam's cross-section
C^0	the class of continuous functions possessing discontinuous derivatives at element nodes
C^{-1}	the class of continuous functions that are discontinuous at element nodes
C_{ij}	elastic stiffness coefficients
D_{ij}	bending rigidities
E_i	elastic moduli
f^e	consistent load vector
G	transverse shear rigidity
$2h$	beam thickness
I_y	cross-sectional moment of inertia about y-axis
K^e	element stiffness matrix
L	beam span
N_x, N_z, Q_x	force resultants
M_x, M_z	moment resultants
q, q^+, q^-	applied transverse loads
S^+, S^-	top and bottom beam surfaces
T_{ij}	prescribed end tractions
u	midplane displacement along x-axis
u_x, u_z	Cartesian displacement components
w, w_i	transverse displacements
x, z	axial and transverse coordinates
γ_{xz}	transverse shear strain
δ	variational operator
ε_{ij}, κ_{ij}	strain and curvature components
θ	bending cross-sectional rotation
η, ξ	dimensionless coordinates
ν_{ij}	Poisson's ratios
σ_{ij}, τ_{xz}	stress components
ℓ	beam element length

CONTENTS

INTRODUCTION

In the simplest cases of beam bending, vibration and stability, analytic solutions can be obtained using the Bernoulli-Euler (classical) or Timoshenko beam theories, with the latter accounting for the deformation due to transverse shear.[1,2] Analytic solutions can be either difficult or impossible to obtain for the many practical applications of beams as reinforcing members in plate and shell structures and whenever nonlinear deformation/material behavior is considered. In all of these cases, the finite element method enables the analyst to obtain sufficiently accurate approximate solutions.

The principal benefit of the Timoshenko beam modeling[3-5] is the ability to properly account for transverse shear deformation, the effect that can be significant in deep beams and those made of laminated composites which are known to exhibit relatively low transverse shear stiffness. Since the Timoshenko theory does not account for the deformation in the transverse normal direction, it precludes solutions to problems which lend themselves to the three-dimensional state of stress. An example is the composite beam subject to impact loading[6] or high-frequency vibrational modes (i.e., short-wavelength loading).

Recently, Tessler[7-9] has developed a higher-order plate theory for linearly elastic orthotropic plates, incorporating the deformation effects due to the inplane stretching, transverse shear and transverse normal straining. The theory includes both linear inplane and quadratic transverse displacement expansions and, due to a special form of a variational statement, is particularly attractive for finite element development. An extension of the above theory to laminated composites and the development of a simple triangular plate element has been presented by Tessler and Saether.[10-11] In this paper, a higher-order theory for linearly elastic orthotropic beams, a one-dimensional analogue of the plate theory,[7-9] is presented. Also, a simple two-node beam finite element is formulated and its predictive capability is evaluated with respect to both analytic and exact elasticity solutions.

In the section on Higher-Order Beam Theory which follows, the essential aspects of the present beam theory are discussed. Its distinguishing feature is that the equations of equilibrium and appropriate boundary conditions are derived from a virtual work theorem which employs both conventional (Timoshenko-type) displacement variables and two higher-order transverse displacement variables. The former variables have the highest spatial derivatives of order one, whereas the latter variables possess no derivatives. The major issue for finite element development is that C^0-continuous finite element approximations being used for the Timoshenko displacements and only (C^{-1}) [1] approximations for the higher-order displacements. The implication is that the higher-order variables can be calculated as auxiliary values and condensed out statically at the finite element level. This results in the element equilibrium equations being of the same complexity as those corresponding to Timoshenko theory elements. The advantage

[1]Whereas C^0 continuous functions are continuous within the element domain and at the nodes, C^{-1} functions are only continuous within the element and can be discontinuous at the nodes. Both of these classes of functions may yield discontinuous derivatives at the nodes.

of the present methodology is that both transverse shear and transverse normal deformations are represented where the resulting equations are the same order as in Timoshenko theory; however, the range of applicability of the present theory allows for extention to thicker beams.

In the Two-Node Beam Element Section, a simple two-node beam element is developed using the present theory. The anisoparametric interpolations,[4,5,12] originally derived to eliminate thin-regime shear locking in Timoshenko beam elements, are used for the element kinematic variables. Both analytic and finite element solutions for a classical beam problem are presented in the Discussion of Results Section and the results are compared with those obtained from Timoshenko and elasticity theories.

HIGHER-ORDER BEAM THEORY

In order to clarify the development of the theory, consider the bending of an elastic orthotropic beam having a narrow rectangular cross-section of width b, height 2h, and spanning the length L; the beam is located in the x-z Cartesian frame with the x-axis ($x \in [0, L]$) passing through the midplane (refer to Figure 1).

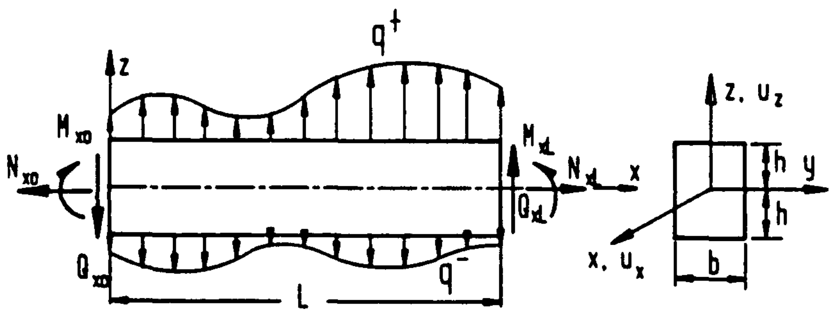

Figure 1. Beam sign convention.

We expand the longitudinal and transverse displacement components (u_x and u_z) with respect to the dimensionless thickness coordinate $\xi = z/h \in [-1, 1]$, where u_z has a special parabolic form:

$$u_x(x, z) = u(x) + h\xi\theta(x), \quad u_z(x, z) = w(x) + \xi w_1(x) + (\xi^2 - \tfrac{1}{5})w_2(x) \tag{1}$$

where $\xi=0$ identifies the position of the reference midplane; note that the expansion for u_z is such that w(x) represents a weighted-average transverse displacement rather than the midplane deflection, i.e.

$$w(x) = \frac{3}{4h}\int_{-h}^{h} u_z(x,z)(1-\xi^2)\,dz \tag{2}$$

The expansion coefficients u(x) and θ(x) are also defined as weighted averages according to

$$u(x) = \frac{1}{2h}\int_{-h}^{h} u_x(x,z)\,dz, \quad \theta(x) = \frac{3}{2h^3}\int_{-h}^{h} u_x(x,z)\,z\,dz \tag{3}$$

where u(x) is the midplane displacement along the x axis, whereas θ(x) is the rotation of the normal about the y axis. Assuming the beam is made of a linearly elastic orthotropic material having its principal material directions coincident with the Cartesian coordinates, a two-dimensional Hooke's law can be written as[13]

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{13} & 0 \\ C_{13} & C_{33} & 0 \\ 0 & 0 & C_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \\ \gamma_{xz} \end{Bmatrix} \tag{4}$$

where the elastic stiffness coefficients C_{ij} can be expressed in terms of the engineering elastic constants as

$$\begin{gathered} C_{11} = \frac{E_1}{1-\nu_{31}\nu_{13}}, \quad C_{33} = \frac{E_3}{1-\nu_{31}\nu_{13}}, \\ C_{13} = C_{33}\nu_{13}, \quad C_{55} = G_{13}, \ and\ E_1\nu_{31} = E_3\nu_{13} \end{gathered} \tag{5}$$

where E_1 and E_3 are the longitudinal and transverse normal elastic moduli, ν_{13} and ν_{31} are the major and minor Poisson ratios, and G_{13} is the transverse shear modulus.

The longitudinal strain-displacement relation has the usual form written as

$$\varepsilon_{xx} = \varepsilon_{x0} + z\,\kappa_{x0} \tag{6}$$

with the beam strains given as

$$\{\varepsilon_{x0},\ \kappa_{x0}\} = \{u(x)_{,x},\ \theta(x)_{,x}\} \tag{6a}$$

where a comma (,) denotes partial differentiation. The transverse normal and transverse shear strains can be expressed according to Reference 7 as

$$\varepsilon_{zz} = \varepsilon_{z0} + \phi_z(\xi)\,\kappa_{z0} + \phi_x(\xi)\,\kappa_{x0}, \quad \gamma_{xz} = \phi_{xz}(\xi)\,\gamma_{xz0} \tag{7}$$

where the beam transverse strain quantities and associated through-thickness shape functions are

$$\begin{aligned} \left\{\varepsilon_{z0},\ \kappa_{z0},\ \gamma_{xz0}\right\} &= \left\{w_1(x)/h,\ w_2(x)/h^2,\ w(x)_{,x} + \theta(x)\right\} \\ \left\{\phi_x,\ \phi_z,\ \phi_{xz}\right\} &= \left\{\tfrac{h}{17}\nu_{13}\xi(4-7\xi^2),\ \tfrac{14}{17}h\xi(3-\xi^2),\ \tfrac{5}{4}(1-\xi^2)\right\} \end{aligned} \tag{7a}$$

The above expansions of the transverse strains constitute a major departure from the conventional displacement formulation. In the present theory, the transverse strains satisfy compatibility in the weak variational sense by

$$\delta\int_{-h}^{h}(\varepsilon_{zz} - u_{z,z})^2 dz = 0, \quad \delta\int_{-h}^{h}(\gamma_{xz} - u_{z,x} - u_{x,z})^2 dz = 0 \tag{8}$$

where δ is the displacement variational operator. These transverse strains also ensure the exact traction conditions at the top and bottom beam surfaces such that

$$\tau_{xz}(x, \pm h) = 0, \quad \sigma_{zz,z}(x, \pm h) = 0 \tag{9}$$

The beam equations of equilibrium together with the natural boundary conditions are obtained from the virtual work principle. Neglecting the body forces, the variational statement may be written as

$$\begin{aligned} &\int_V (\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{zz}\delta\varepsilon_{zz} + \tau_{xz}\delta\gamma_{xz})dA\,dx - \int_{S^+} q^+(x)\delta u_z(x, h)dxdy - \int_{S^-} q^-(x)\delta u_z(x, -h)dxdy \\ &+ \int_A [T_{x0}\delta u_x(0, z) + T_{z0}\delta u_z(0, z)]dA - \int_A [T_{xL}\delta u_x(L, z) + T_{zL}\delta u_z(L, z)]dA \ = \ 0 \end{aligned} \tag{10}$$

where S^+ and S^- denote the top and bottom beam surfaces that are free of shear tractions and subject to the transverse pressure loads

$$\begin{aligned} \tau_{xz}(x,\ h) &= 0, \quad \sigma_{zz}(x,\ h) = q^+(x) \quad on\ S^+ \\ \tau_{xz}(x, -h) &= 0, \quad \sigma_{zz}(x, -h) = q^-(x) \quad on\ S^- \end{aligned} \tag{11}$$

where T_{i0} and T_{iL} (i=x, z) are the tractions prescribed at the two ends of the beam (x=0, L), and A is the beam's cross-sectional area (see Figure 1).

Integrating Equation 10 over the cross-sectional area results in a one-dimensional virtual work statement written as

$$\begin{aligned} &\int_0^L (N_x\delta\varepsilon_{x0} + N_z\delta\varepsilon_{z0} + M_x\delta\kappa_{x0} + M_z\delta\kappa_{z0} + Q_x\delta\gamma_{xz0})dx - \int_0^L \{\overline{q_1}(\delta w + \tfrac{4}{5}\delta w_2) + \overline{q_2}\delta w_1\}dx \\ &+ N_{x0}\delta u(0) + M_{x0}\delta\theta(0) + Q_{x0}\delta w(0) - N_{xL}\delta u(L) - M_{xL}\delta\theta(L) - Q_{xL}\delta w(L) \ = \ 0 \end{aligned} \tag{12}$$

where the stress resultants are defined as

$$(N_x, N_z) = \int_A (\sigma_{xx}, \sigma_{zz})dA, \quad (M_x, M_z) = \int_A (z\sigma_{xx}, \phi_z\sigma_{zz})dA, \quad Q_x = \int_A \phi_{xz}\tau_{xz}dA, \tag{13}$$

(*A is the cross-sectional area*)

and the applied normal tractions, end forces and moments are represented by

$$(q_1, q_2) = (q^+ - q^-, q^+ + q^-), \quad (\overline{q_1}, \overline{q_2}) = (q_1 b, q_2 b) \tag{14}$$

$$(N_{x0}, N_{xL}) = \int_A (T_{x0}, T_{xL})dA, \quad (M_{x0}, M_{xL}) = \int_A (T_{x0}, T_{xL})z\,dA, \quad (Q_{x0}, Q_{xL}) = \int_A (T_{z0}, T_{zL})dA$$

where T_{z0} and T_{zL} have the same parabolic distribution across the beam thickness as the shear stress, τ_{xz}.[7]

In Equation 12, expressions associated with the arbitrary variations δw_1 and δw_2 must vanish independently, thus yielding the two higher-order transverse equilibrium equations[2]

$$N_z/h - \overline{q_2} = 0, \quad M_z/h^2 - 4\overline{q_1}/5 = 0 \tag{15}$$

Integrating Equation 12 by parts results in the remaining equilibrium equations

$$N_{x,x} = 0, \quad M_{x,x} - Q_x = 0, \; and \; Q_{x,x} + \overline{q_1} = 0 \tag{16}$$

with the natural boundary conditions:

At x=0:

$$N_x(0) = N_{x0} \; or \; \delta u(0) = 0, \quad M_x(0) = M_{x0} \; or \; \delta\theta(0) = 0, \quad Q_x(0) = Q_{x0} \; or \; \delta w(0) = 0 \tag{17}$$

At x=L:

$$N_x(L) = N_{xL} \; or \; \delta u(L) = 0, \quad M_x(L) = M_{xL} \; or \; \delta\theta(L) = 0, \quad Q_x(L) = Q_{xL} \; or \; \delta w(L) = 0 \tag{18}$$

Computing the stress resultants (Equation 13) by applying both Hooke's law (Equation 4) and the strain expressions (Equations 6 and 7) results in the beam constitutive relations which can be expressed in matrix form as

$$\begin{Bmatrix} N_x \\ N_z \\ M_x \\ M_z \\ Q_x \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{13} & 0 & 0 & 0 \\ A_{13} & A_{33} & 0 & 0 & 0 \\ 0 & 0 & D_{11} & D_{13} & 0 \\ 0 & 0 & D_{13} & D_{33} & 0 \\ 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{z0} \\ \kappa_{x0} \\ \kappa_{z0} \\ \gamma_{xz0} \end{Bmatrix} \tag{19}$$

---

[2]At this stage there is no integration by parts since $w_1(x)$ and $w_2(x)$ have no spatial derivatives in the variational statement (Equation 12).

where the A_{ij}, D_{ij} and G beam rigidities are given as

$$A_{11} = C_{11}A, \quad A_{13} = C_{13}A, \quad A_{33} = C_{33}A,$$

$$D_{11} = \left(C_{11} - \frac{C_{13}^2}{85C_{33}}\right)I_y, \quad D_{13} = \frac{168}{85}I_yC_{13}, \tag{20}$$

$$D_{33} = \frac{336}{85}I_yC_{33}, \; and \; G = k^2AG_{13}$$

where

$$k^2 = \frac{5}{6} \quad and \quad I_y = \int_A z^2 dA \tag{20a}$$

For a rectangular cross-section

$$A = 2hb \; and \; I_y = \frac{2}{3}h^3b \tag{20b}$$

Substituting the relations for N_z and M_z from Equation 19 into Equation 15 yields the transverse normal equilibrium equations in terms of displacements which in nondimensional form appear as

$$\begin{aligned} \frac{w_1}{h} + \nu_{13}u_{,x} &= \frac{q_2}{2C_{33}} \\ \frac{w_2}{h} + \frac{\nu_{13}}{2}h\theta_{,x} &= \frac{17q_1}{56C_{33}} \end{aligned} \tag{21}$$

which are readily solved for w_1 and w_2. Introducing these variables into Equation 19 and then substituting the force and moment resultants into Equations 15 results in the remaining equilibrium equations written in terms of displacements as

$$\begin{aligned} (E_1Au_{,x})_{,x} + (\nu_{13}h\overline{q_2})_{,x} &= 0 \qquad (a) \\ (E_1I_y\theta_{,x})_{,x} + \tfrac{2}{5}(\nu_{13}h^2\overline{q_1})_{,x} - k^2AG_{13}(w_{,x} + \theta) &= 0 \qquad (b) \\ [k^2AG_{13}(w_{,x} + \theta)]_{,x} + \overline{q_1} &= 0 \qquad (c) \end{aligned} \tag{22}$$

As expected from a linear (i.e., small displacement) theory, the stretching (Equation 22a) and bending (Equations 22b and 22c) equilibrium equations are uncoupled. Observe that by neglecting the Poisson effect (i.e., by setting ν_{13}=0), Equations 22 can be reduced to those of Timoshenko theory. Note, even though a higher-order transverse displacement expansion governs the deformation in the present theory, the boundary conditions (Equation 18) are the same as in Timoshenko theory.

In the present theory, the solution procedure involves integrating Equation 22 subject to the boundary conditions given by Equations 17 and 18, then substituting u(x) and $\theta(x)$ into Equation 21 to obtain $w_1(x)$ and $w_2(x)$. Note, the stress components are obtained in a consistent manner from Hooke's law (Equation 4).

A TWO-NODE BEAM ELEMENT

To demonstrate the computational suitability of the present theory, a simple two-node beam element is derived directly from the displacement variational principle (Equation 12). The interpolation requirements for the element displacement field stem from Equation 12, where, as in Timoshenko theory, the weighted-average displacements u(x), w(x) and $\theta(x)$ have derivatives that do not exceed order one; therefore, C^0 shape functions can be used. Equation 12 contains no spatial derivatives of $w_1(x)$ and $w_2(x)$, therefore the variables need only be C^{-1} continuous (i.e., discontinuous at the element nodes).

Now consider a two-node beam element of unit width (b=1), span ℓ and height 2h, loaded at the top surface by a uniform distributed pressure $q^+(x)=q$. The displacement interpolations can be expressed directly in terms of nodal coordinates as[4,5] which in the present notation are superscribed with an ℓ, i.e.,

<u>C^0 -Continuous, Linear Functions</u>

$$u(x) = (1-\eta)u^\ell_0 + \eta u^\ell_1, \quad \theta(x) = (1-\eta)\theta^\ell_0 + \eta\theta^\ell_1 \tag{23a}$$

<u>C^0 -Continuous, Quadratic Functions</u>

$$w(x) = (1-\eta)w^\ell_0 + \eta w^\ell_1 + \frac{\ell}{2}\eta(1-\eta)(\theta^\ell_0 - \theta^\ell_1) \tag{23b}$$

<u>C^{-1} -Continuous, Uniform Function</u>

$$w_1(x) = W^\ell_1, \quad w_2(x) = W^\ell_2 \tag{23c}$$

where $\eta = x/\ell \in [0, 1]$, and u^ℓ_i, θ^ℓ_i and w^ℓ_i (i=0, 1) denote nodal degrees-of-freedom (dof). Note that Equations 23 yield the beam strains that are uniform across the element span

$$\begin{aligned} \gamma_{xz0} &= \frac{1}{\ell}(w^\ell_1 - w^\ell_0) - \frac{1}{2}(\theta^\ell_0 + \theta^\ell_1) && (a) \\ (\varepsilon_{x0}, \kappa_{x0}) &= \frac{1}{\ell}(u^\ell_1 - u^\ell_0, \quad \theta^\ell_1 - \theta^\ell_0) && (b) \\ (\varepsilon_{z0}, \kappa_{z0}) &= \frac{1}{h}(W^\ell_1, \quad W^\ell_2/h) && (c) \end{aligned} \tag{24}$$

Substituting Equations 24 and 19 into the variational principle (Equation 12) yields the element stiffness equilibrium equations in terms of the six engineering dof (u^ℓ_i, w^ℓ_i, θ^ℓ_i, where i= 0, 1) and the two higher-order displacements, W^ℓ_1 and W^ℓ_2. Because the latter variables are discontinuous at the nodes, they are readily eliminated at the element equilibrium level through static condensation resulting in the following

$$W^{\ell}_1 = h\left[\frac{q}{2C_{33}} - \frac{\nu_{13}}{\ell}(u^{\ell}_1 - u^{\ell}_0)\right], \quad W^{\ell}_2 = h\left[\frac{17q}{56C_{33}} - \frac{\nu_{13}h}{2\ell}(\theta^{\ell}_1 - \theta^{\ell}_0)\right] \tag{25}$$

Note that the same result can be obtained from the exact equilibrium equations (Equation 21) by simply substituting the element displacement interpolations (Equation 23).

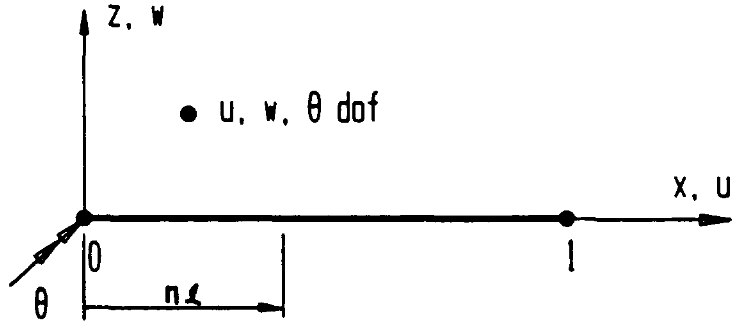

Figure 2. A two-node stretching-bending beam element.

The resulting two-node element is capable of axial stretching and bending accounting for both transverse shear and transverse normal deformations. The element has the simplest nodal pattern with six engineering dof (see Figure 2).

The element stiffness matrix K^e and the consistent load vector f^e corresponding to the vector of nodal displacement dof $\{u_0, u_1, w_0, w_1, \theta_0, \theta_1\}^t$ have the form

Stiffness Matrix

$$K^e = \begin{bmatrix} k_{11} & k_{12} & 0 & 0 & 0 & 0 \\ k_{12} & k_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{33} & k_{34} & k_{35} & k_{36} \\ 0 & 0 & k_{34} & k_{44} & k_{45} & k_{46} \\ 0 & 0 & k_{35} & k_{45} & k_{55} & k_{56} \\ 0 & 0 & k_{36} & k_{46} & k_{56} & k_{66} \end{bmatrix} \tag{26}$$

where

$$k_{11} = k_{22} = -k_{12} = \frac{2h}{\ell}E_1, \quad k_{33} = k_{44} = -k_{34} = \frac{2hk^2G_{13}}{\ell}$$

$$k_{35} = k_{36} = -k_{45} = -k_{46} = \frac{\ell}{2}k_{33}, \quad k_{55} = k_{66} = \frac{1}{\ell}\left(\frac{2h^3}{3}E_1 + \frac{\ell^2}{2}hk^2G_{13}\right) \tag{26a}$$

$$k_{56} = \frac{1}{\ell}\left(-\frac{2h^3}{3}E_1 + \frac{\ell^2}{2}hk^2G_{13}\right)$$

<u>Consistent Load Vector</u>

$$f^e = \{f_1, -f_1, f_2, f_2, f_3, -f_3\}^T \tag{27}$$

where

$$f_1 = q\nu_{13}h, \quad f_2 = \tfrac{1}{2}q\ell, \quad f_3 = q\left(\tfrac{2}{5}h^2\nu_{13} - \tfrac{1}{12}\ell^2\right) \tag{27b}$$

The above element stiffness matrix (Equation 26) is identical to that of the two-node element derived from Timoshenko theory;[4] however, the consistent load vector (Equation 27) involves the Poisson ratio ν_{13} which does not appear in the Timoshenko element. (Note that in Reference 4, the axial deformations have not been considered.) Since ν_{13} appears in f_1, it follows that the element is capable of predicting the deformation of the midplane x-axis due to the transverse load q, which is consistent with the theory of elasticity but is not accounted for in Timoshenko theory. As in Timoshenko theory, the behavior of the element in the thin regime (as $h \to 0$) is governed by the ability of the kinematic field to accommodate the Kirchhoff constraint of the vanishing transverse shear as follows

$$\gamma_{xz0} = \frac{1}{\ell}(w^e_1 - w^e_0) - \tfrac{1}{2}(\theta^e_0 + \theta^e_1) \to 0 \tag{28}$$

Moreover, in this theory, the inextensibility of the transverse normal fiber is also enforced in the thin limit

$$(w_1, w_2) \to 0 \tag{29}$$

In the Kirchhoff constraint (Equation 28), the w^e_i and θ^e_i (i= 0, 1) dof are consistently balanced. This is generally sufficient to guarantee locking-free element performance in the thin regime. However, shear locking can still take place when excessive kinematic restraints are enforced on a single element, such as in the case of a fixed-pinned beam;[9] e.g., when the displacement boundary conditions are set $w^e_0=\theta^e_0=w^e_1=0$, giving rise to $\theta^e_1 \to 0$ from Equation 28, which in turn yields zero bending curvature $\kappa_{x0} \to 0$ according to Equation 24, or what is known as shear locking. This pathological case, however, is readily resolved when two or more elements are used in the finite element discretization. To further improve element performance and to eliminate shear locking from the modeling case just described, a shear relaxation parameter (a device which consistently relaxes the Kirchhoff constraint at the element level) can be effectively employed.[5,9]

DISCUSSION OF RESULTS

To assess the predictive capability of the higher-order beam theory and to examine the behavior of the proposed two-node beam element, a simple bending problem involving a simply-supported, rectangular cross-section isotropic beam under uniform loading was considered. This problem, having an exact two-dimensional elasticity solution,[14] encompasses the effects of both transverse shear and transverse normal deformations; therefore, representing a means for evaluating our higher-order beam theory.

The analytic solution according to the present theory is obtained by introducing the uniformly distributed loading q (applied at $\xi=1$) into the differential equations of equilibrium (Equation 22), which are then solved for the weighted-average displacement variables u(x), w(x), and $\theta(x)$ satisfying the boundary conditions

$$\begin{aligned} u(0) = w(0) = M_x(0) = 0 \\ w(L) = N_x(L) = M_x(L) = 0 \end{aligned} \qquad (30)$$

The beam is taken to be of unit width (b=1), thickness 2h, and it spans $x \in [0, L]$.

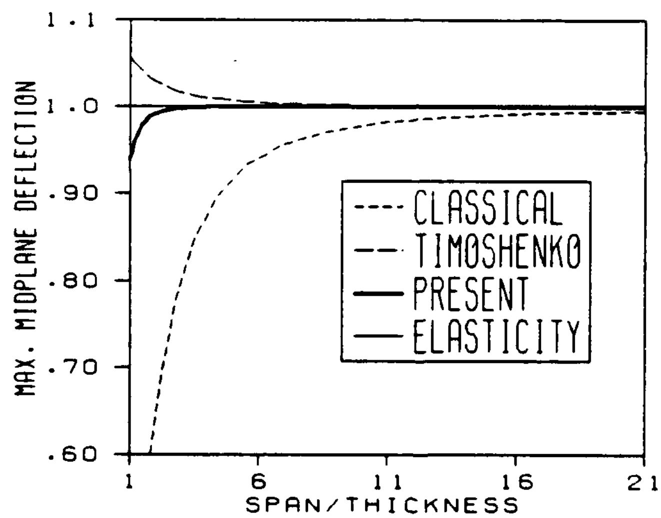

Figure 3. Maximum midplane deflection versus span-to-thickness ratio.

In Figures 3 and 4, the analytic solutions of the present higher-order beam theory are compared with those of the classical and Timoshenko theories, as well as the exact elasticity solution. Figure 3 depicts the maximum midplane deflection $u_z(L/2,0)$, normalized with respect to the exact elasticity solution, which is plotted versus the span-to-thickness ratio (L/2h). It is seen that even in the extreme thickness case of L/2h=1, the present theory underestimates the maximum midplane deflection only by about 5%. Although the deflection curve due to Timoshenko theory is presented for comparison, it only represents a weighted-average deflection and not the

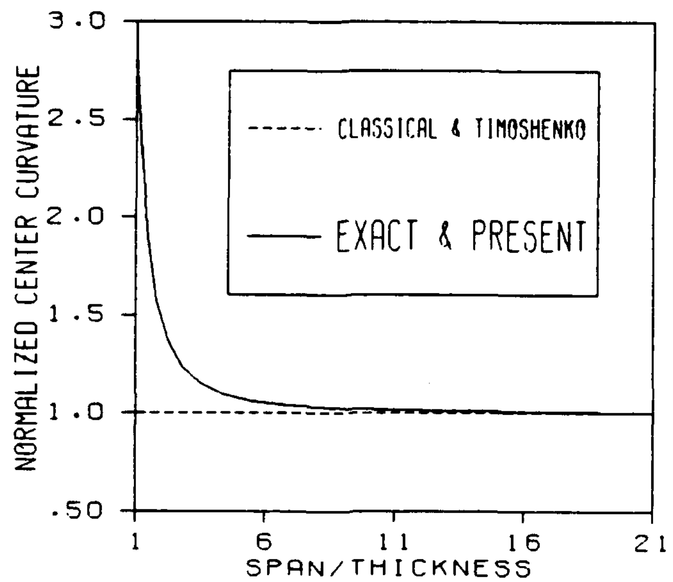

Figure 4. Center-span curvature versus span-to-thickness ratio.

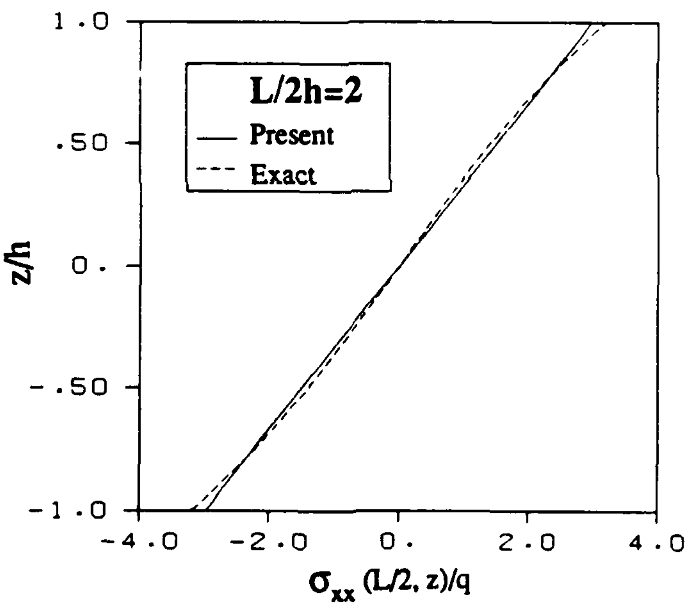

Figure 5. Distribution of axial stress across thickness for deep beam (L/2h=2).

midplane deflection, where the latter quantity is unobtainable from the Timoshenko theory. Figure 4 shows a normalized value (with respect to the classical solution) of the curvature computed at the center the beam span (x=L/2) versus L/2h. Both the exact elasticity and the present higher-order theory show identical solutions for the entire range of L/2h, whereas the classical and Timoshenko theories produce appreciably erroneous results in the thick regime (for L/2h≤5). This result provides evidence for the acceptability of the higher-order theory.

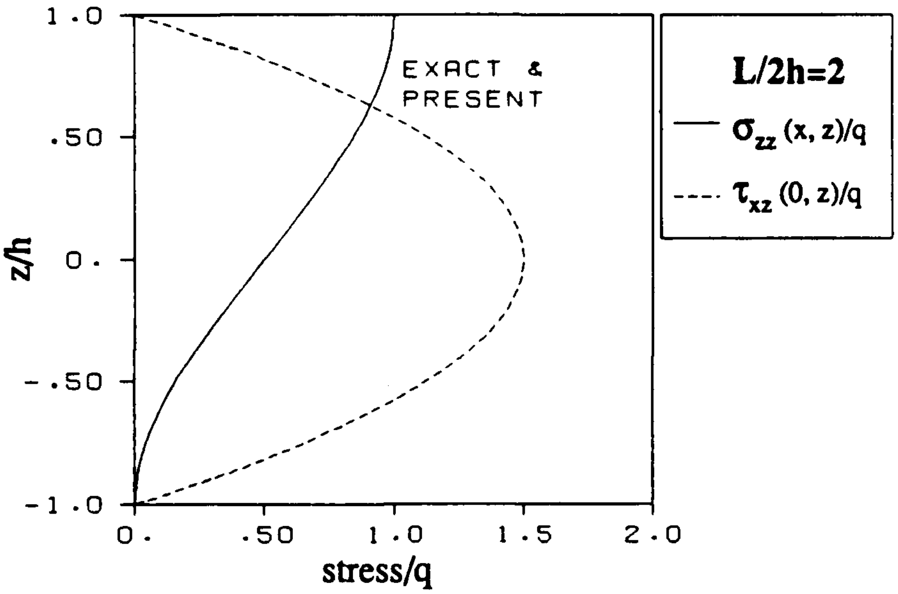

Figure 6. Distributions of transverse shear and transverse normal stresses across thickness for deep beam (L/2h=2).

Figures 5 and 6 demonstrate the through-thickness distributions of the stress components $\sigma_{xx}(L/2,z)$, $\sigma_{zz}(x,z)$ and $\tau_{xz}(0,z)$ for the case of a very deep beam, L/2h=2. Whereas rather slight differences are observed in the prediction for the normal stress (Figure 5), the transverse stresses obtained from the present theory agree with those of the exact solutions (Figure 6).

The results obtained by the finite element discretization, employing regular meshes with the use of the two-node beam element, are shown in Figures 7 through 9, where convergence studies for the displacement, strain and stress variables for the deep beam case (L/2h=2) are presented. The errors were computed from comparisons with appropriate exact elasticity solutions. Throughout, the results are either exact or converge rapidly to the exact solutions with mesh refinement.

Finally, it should be noted that the present two-node element, having the same stiffness matrix as its Timoshenko theory counterpart,[4] does not suffer from shear locking in the thin regime. The element can also be used with the shear relaxation parameter[5] to further improve results for coarsely discretized models.

CONCLUDING REMARKS

This report presented a variational higher-order theory for the bending and stretching of elastic orthotropic beams, including both transverse shear and transverse normal deformations. The particular appeal of the theory is that it provides a displacement variational framework for developing effective and computationally efficient beam finite elements with the ability to predict accurately the through-thickness distributions of all displacement, strain and stress components.

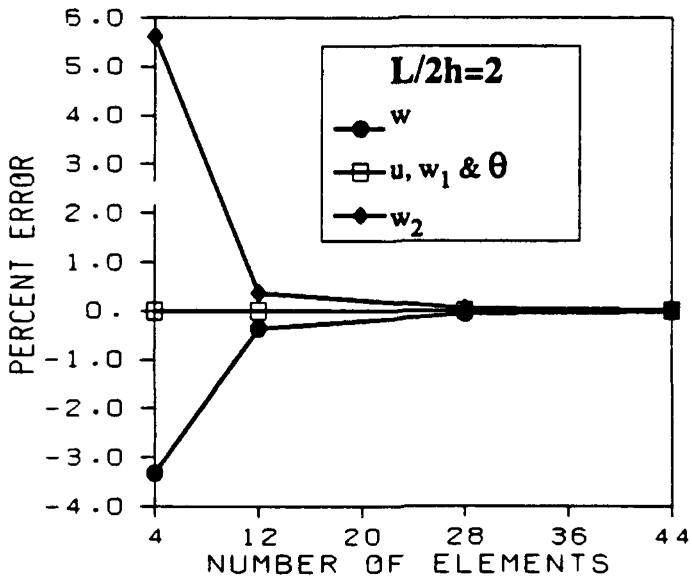

Figure 7. Convergence of maximum kinematic variables for deep beam (L/2h=2).

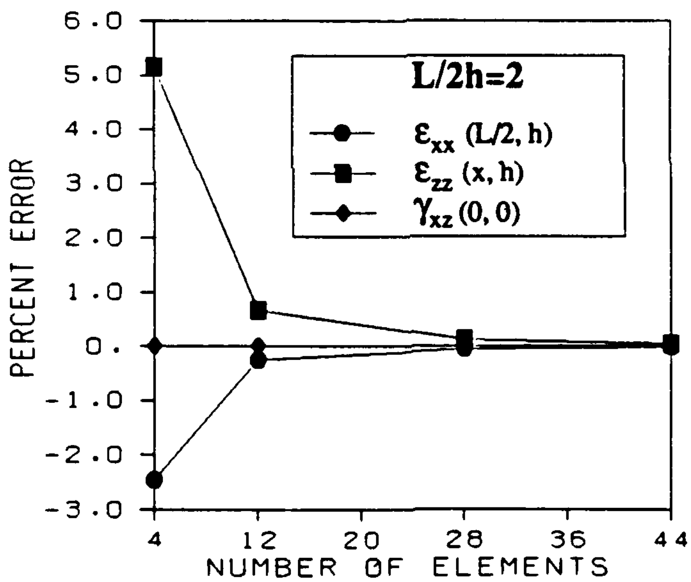

Figure 8. Convergence of maximum strains for deep beam (L/2h=2).

A simple two-node element, derived from the variational theorem, demonstrated improved modeling capabilities over a comparable Timoshenko element, particularly, in the analysis of thick-section beams.

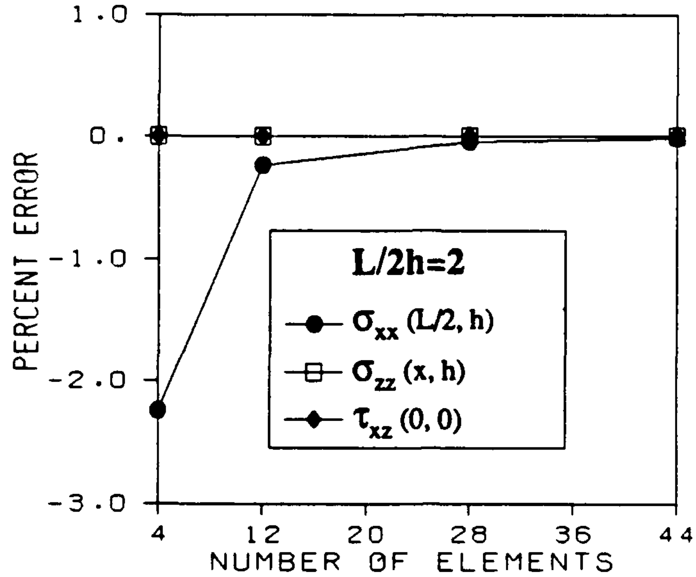

Figure 9. Convergence of maximum stresses for deep beam (L/2h=2).

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